OpenRisk White Paper

Revisiting simple concentration indexes

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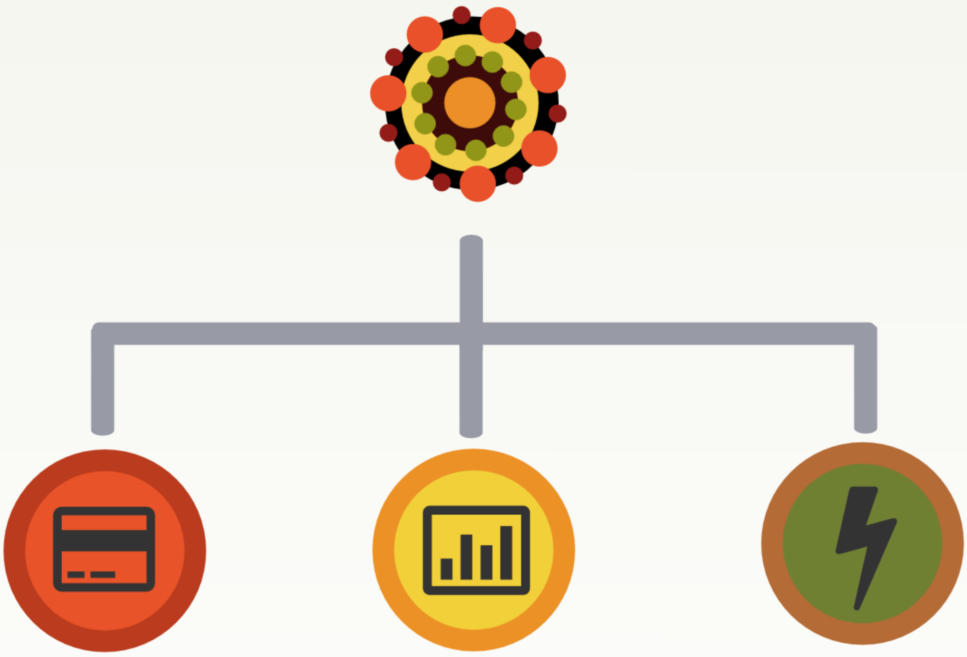
March 27, 2015



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SUMMARY

We review the definitions of widely used concentration metrics such as the concentration ratio, the HHI index and the Gini and clarify their meaning and relationships. This new analytic framework helps clarify the apparent arbitrariness of simple concentration indexes and brings to the fore the underlying unifying concept behind these metrics, thereby enabling their more informed use in portfolio and risk management applications. We also propose that the sensitivity of concentration indexes to growing concentration should be a defining criterion for adopting an index and explore the sensitivity of common indexes to changing portfolio concentrations. We show that this sensitivity can vary significantly between indexes for parametric families of portfolio distributions and hence selecting and using a simple concentration index should take this aspect carefully into consideration.
The white paper has three sections:

- A *concept* section discussing the issues and the proposal in non-technical terms,
- a *technical paper* offering precise definitions and numerical studies and
- an *open source* implementation section

Further Resources

The OpenRisk Academy offers a range of online courses around credit concentration which utilize the latest in interactive eLearning tools, for example:

- Introduction to Name Concentration Measurement,
- Introduction to Sector Concentration Measurement,
- Credit Concentration Add-Ons in the UK Pillar II methodology

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Concept Paper

Credit concentration in asset portfolios is an important example of risk concentration. It arises from imperfect diversification of ”idiosyncratic” or ”specific” risks arising from small portfolio size or unusually large exposures to individual clients (counterparties), sectors, products, sovereigns or regions.
Over the course of the years, a wide variety of methods have been developed to assist with the quantification of credit concentration risk. For a broad overview of the state of the art see [1].
It is now customary to distinguish between so-called *ad-hoc* (also called model-free) methods such as the famous *Herfindahl-Hirschman* index versus methods based on *credit portfolio models* (elso called economic capital models). Ad hoc measures generally use only exposure data. While such data may still be derived from other models (e.g. EAD models) there is the implicit notion that the ”real” risk model content (and model risk) comes from the use of data such as default probabilities and correlations between credit risks. Therefore, model-based approaches are considered to have a higher informational value and in the past decade there has been an almost exclusive focus on credit portfolio models (or portfolio model approximations) as the primary means for quantifying credit concentrations.
Ad-hoc measures are a standard part of the regulatory toolkit (for example Working Paper No 23, ”The Proposed Revised Ratings Based Approach” for Securitisation and references to therein for identifying a ”granular pool” of securitised assets and also in the context of Solvency II, Joint Forum Stocktaking on the use of credit ratings Jun 2009. We will not attempt an exhaustive review of the use of such metrics as several relevant references already provide this [2, 1, 3].
The original Basel II consultative document included the HHI concentration index as input into the so called granularity adjustment (was eventually dropped)[4]. The HHI indicator is quite flexible and has been used to measure name, asset or geographic concentration[5]. A review of concentration indicators and methodologies applicable when using aggregate data are discussed in Avila [6].

Pros and Cons of the two approaches

While the criticism of simple approaches is entirely valid, it is very instructive to try to understand the ”ad-hoc” indices as they are not without significant advantages:

- They are based on simple formulas and thus offer consistent and transparent estimates across portfolios and firms
- Their simplicity leads also to practical (e.g., easy and fast) calculations
- Their lack of risk sensitivity can, in-principle, be remedied by using risk adjusted exposures (e.g., expected loss)

At the same time, while modern credit portfolio modeling frameworks are in principle a powerful tool for tackling issues of credit risk concentration they have a number of significant weaknesses:

- There is a variety of possible models with potential for wide variability of results (model risk)
- With the possible exception of name concentration, it is neither obvious nor has there been adequate exploration and standardization of concentration measure extraction from a "full" model
- Portfolio models are significantly more difficult to implement and use

Given this rather inconclusive state of affairs, in this white paper we revisit and explore the so-called ad-hoc measures with a view to help clarify their meaning and fitness for use.

Ad-hoc concentration measures

Our focus is instead on the *interpretation* and relationships. This aspect is decidedly less developed and understood in credit risk context. We will discuss specifically the linkages between the following:

Box 1. Commonly used concentration indexes

- The simple Concentration Ratio (based on the fractional size or large exposures)
- The famous Herfindahl-Hirschman index (sum of squared exposure weights)
- The equally famous Gini index (based on the rank ordering of exposures)
- The more technical Shannon and Hannah-Kay indices

These concentration indices are called "ad-hoc" because of three primary reasons:

- They do not use all available risk information. Hence there is a somewhat arbitrary focus on exposure data, although quite obviously exposure is a key driver of concentration
- There are no intrinsically defined *benchmark values* for the concentration thresholds of each index. Despite the long use in policy work, the "calibration" of what constitutes a concentrated portfolio is essentially a convention
- As we saw, there are multiple possible indexes, with unclear relationships between the different measures.

The relationship of the different ad-hoc measures

To unify the different concentration metrics such as the concentration ratio, the HHI, the Gini etc. we recast what these indices measure by tracking what happens to a *small unit exposure*. To do that, we first sort our exposures from largest to smallest.
The two key "outcomes" we track are the following:

- If we pick a unit exposure (say a single euro/dollar) at random, which exposure is our single euro part of? That is, what is its rank ordering?
- How large is the exposure to which our random unit belongs to?

> **Box 2. Key concept**
> Consider a single exposure E (say 100 mln EUR to counterparty A). Calculate its fraction of the total portfolio exposure E_T (say 10 bln total). This fraction (1 %) is also the *probability* that a unit of exposure (e.g. a single euro) drawn at random from the portfolio would be part of the exposure to counterparty A.

While quite simple, this concept allows us to express all the various concentration indices in a probabilistic language of expectations:

> **Box 3. Re-interpretation of the various concentration indices**
> - The concentration ratio captures the probability that a euro of exposure is part of the largest k exposures
> - The HHI index captures the *expected size* of the exposure to which a euro exposure belongs
> - The Gini index captures the *expected rank order* of the exposure to which a euro exposure belongs
> - The Shannon index captures the number of ways we can arrange unit exposures to construct our actual portfolio distribution

Other less used measures such as the Hannah-Kay family are also integrated. This unified language suggests that the different ad-hoc indices are essentially just different ways of assessing the *likelihood* of having a relatively large exposure.

Which index to choose?

While were able to rationalize the variety of ad-hoc concentration indexes as expectations of either exposure or rank-order value, the framework does not allow in itself to select some of these measures as the preferred means for assessing concentration. A set of criteria that over time has being established as a desirable list is discussed e.g., in in [7, 8, 9, 1] and is indicatively listed here:

1. The reduction of a loan exposure and an equal increase of a bigger loan must not decrease the concentration measure (transfer principle).
2. The measure of concentration attains its minimum value, when all loans are of equal size (uniform distribution principle).
3. If two portfolios, which are composed of the same number of loans, satisfy that the aggregate size of the k biggest loans of the first portfolio is greater or equal to the size of the k biggest loans in the second portfolio for $1 \leq k \leq N$, then the same inequality must hold between the measures of concentration in the two portfolios (Lorenz-criterion).

4. If two or more loans are merged, the measure of concentration must not decrease (superadditivity).

5. Consider a portfolio consisting of loans of equal size. The measure of concentration must not increase with an increase in the number of loans (in- dependence of loan quantity).

6. Granting an additional loan of a relatively low amount does not increase the concentration measure. More formally, if s denotes a certain percentage of the total exposure and a new loan with a relative share of sn s of the total exposure is granted, then the concentration measure does not increase (irrelevance of small exposures).

Testing the standard indices against these criteria shows that they are indeed generally satisfied, which *does not help to select which metric to use*.

The problem of selecting the ”best” metric to capture the concentration of a portfolio distribution is actually too abstract (not well defined). To achieve more definiteness and narrow down choices we must set a context, in particular around the typical nature of the portfolio distribution and possibly also the intended use of the index (e.g. for different tasks around risk and capital management).

The *sensitivity* of a concentration index to a change in large exposures is an important feature that should enhance its suitability as a warning signal for a continuously monitored portfolio. Sensitivity to ”large exposures” is not easily defined for a completely general portfolio distribution. In the technical part we explore the behavior of concentration indexes for representative families of distributions which, while not the general case, provide valuable insights.

Box 4. Conclusions on the sensitivity of concentration indexes

- The concentration ratio and the Gini are more sensitive indices
- The HHI index is a less sensitive index
- The Shannon index and Hannah-Kay indexes offer intermediate performance (in terms of sensitivity)

Technical Paper

A unified probabilistic interpretation of concentration measures

In this section we recast several familiar concentration risk measures and concepts in a unified probabilistic language. Our core mathematical object is the distribution of n "exposures" $E_i, i \in [1, n]$. An "exposure" can encode any of a number of relevant portfolio data: notional amounts, exposure at default (EAD), potential future exposure (PFE), risk adjusted exposures that use credit spread or probability of default information etc. The main requirement is that the exposures admit a meaningful *summation* into a *total portfolio exposure*:

$$E_T = \sum_{i=1}^{n} E_i \tag{1}$$

This requirement restricts the exposure indicators E_i to be of a *numerical type* instead e.g., of nominal or ordinal type. We also require exposures to be scalar values instead of vectors.
The summation operation allows the definition of exposure *weights*:

$$w_i = \frac{E_i}{E_T} \tag{2}$$

These weights can be interpreted as *probabilities* because they satisfy

$$\sum_{i=1}^{n} w_i = 1 \tag{3}$$

$$w_i \in [0, 1] \tag{4}$$

The elementary event in this probability space is the selection at random of a *small portfolio exposure* dE out of the total portfolio E_T exposure and the identification of the exposure $E \in \{E_i\}$ that it belongs to. The sample space is composed of the collection of all exposures $\{E_i\}$, i.e., a fractional exposure is always part of one actual exposure with probability w_i. For distinct exposures E_i and E_j, the probability that two infinitesimal exposures picked at random belong to one or the other is simply the sum of the weights $w_i + w_j$. A related random variable that will be very useful is the index (rank-order) I of the exposure to which a unit exposure belongs to, i.e., $I \in [1, n]$
In summary, we introduced the two random variables (E, I) which are the outcomes of sampling a small portfolio exposure and identifying the ranking order and size of the exposure it belongs to. While fairly straightforward, if not elementary, this recasting of portfolio weights as probabilities allows for a uniform interpretation of the various concentration metrics and does not seem widely known. We will show that most of the commonly used concentration indexes are indeed *expectations* of (E, I) under the measure we just introduced.

Concentration Ratio and the Cumulative Distribution Function

In order to define the standard concentration ratio CR_k [1] we need to assume that the exposures E_i are *sorted* by size (this requirement will also apply to the Gini index). This ordering is always possible, as we assumed exposures have a scalar numerical type but it does require introducing more structure, namely the probability distribution after the sorting is a *discrete monotone distribution* as opposed to general discrete probability distribution. Given that the index i of the exposures does not have any other intrinsic meaning besides the identification of exposures, we actually have no loss of generality. Despite this, it is maybe worth noting that the sorting of exposures is *not* required for the HHI and Shannon index families we discuss below.
For definiteness we fix a *decreasing* order of probability weight, that is

$$i < j \implies w_i \geq w_j \tag{5}$$

The definition of the **concentration ratio of order** k is simply the sum of the first k weights:

$$\mathrm{CR}_k = \sum_{i=1}^{k} w_i \tag{6}$$

The probabilistic interpretation is the likelihood that a unit exposure picked at random will belong to the first (largest) k exposures:

$$\mathrm{CR}_k = \Pr(E \in \{E_k\}) = \Pr(I \leq k) = \mathrm{CDF}(k) \tag{7}$$

Hence the concentration ratio is simply the CDF of the (sorted) portfolio distribution. In this probabilistic language a concentrated portfolio (with high CR_k) will have a high likelihood that a unit exposure drawn at random will be part of a particularly large position. The range of CR_k, $(0 \geq CR_k \leq 1)$ is simply a property of the CDF function.

Expected Exposure and the HHI

For a given portfolio, if we sample an infinitesimal exposure dE, then the *expected exposure* $\bar{E}$ it belongs to is given as usual by

$$\bar{E} = \mathbb{E}(E) = \sum_{i=1}^{n} w_i E_i = E_T \sum_{i=1}^{n} w_i^2 \,. \tag{8}$$

In our notation the standard Herfindahl-Hirschman Index [1] is then expressed as

$$\mathrm{HHI} = \sum_{i=1}^{n} w_i^2 = \frac{\bar{E}}{E_T} \tag{9}$$

Thus **the HHI is interpreted as the ratio of the expected exposure $\bar{E}$ over the total portfolio exposure E_T**. The higher the concentration as measured by the HHI, the higher the expected exposure when drawing a unit exposure at random.
It might be useful to contrast the expected exposure as defined here with the *average portfolio exposure*

$$E_A = \sum_{i=1}^{n} \frac{1}{n} E_i \tag{10}$$

The latter expresses the expected exposure *under a different measure*, namely picking one of the E_i *exposures* at random and not picking a *unit* exposure at random.

The largest value for the HHI (namely unity) is when the expected exposure coincides with the total exposure, which happens when there is only one exposure with unit probability. The lowest value for the HHI (namely 1/n) is when the expected exposure is equal to any of the equal individual exposures.

For comparability with other indices, we will use the scaled HHI index

$$\mathrm{HHI}' = \frac{\mathrm{HHI} - \frac{1}{n}}{1 - \frac{1}{n}} \tag{11}$$

which ranges from zero for a uniform distribution to unity for a single exposure portfolio.

Higher Moments and the Hannah-Kay index

The higher moments of the distribution of E are given by

$$\mathbb{E}(E^a) = \sum_{i=1}^{n} w_i E_i^a = E_T^a \sum_{i=1}^{n} w_i^{a+1} \tag{12}$$

The generalized Hannah Kay index is defined as [8]

$$\mathrm{HK}_a = (\sum_{i=1}^{n} w_i^a)^{1/(1-a)} \tag{13}$$

For comparability we use the inverted version

$$\mathrm{HKI}_a = (\sum_{i=1}^{n} w_i^a)^{1/(a-1)} \tag{14}$$

which is written in terms of the moment as

$$\mathrm{HKI}_{a+1} = \frac{\mathbb{E}(E^a)^{1/a}}{E_T} \tag{15}$$

which we further normalize so

$$\mathrm{HKI}_a' = \frac{\mathrm{HKI}_a - \frac{1}{n}}{1 - \frac{1}{n}} . \tag{16}$$

Obviously the HHI is the special case for $a = 1$.

The Gini index

The Gini index is defined as

$$G = \frac{1}{n} \sum_{i=1}^{n} (1 - 2i) w_i + 1 \tag{17}$$

If we define the *expected exposure index*

$$\bar{I} = \mathbb{E}(I) = \sum_{i=1}^{n} i w_i \tag{18}$$

as the expected index of a random unit exposure, then we see that in our language the Gini index is related to $\bar{I}$.

$$G = 1 + \frac{1 - 2\bar{I}}{n} \tag{19}$$

For uniform portfolios the expected index $\bar{I}$ takes its midway value (n+1)/2 and hence the Gini is zero. For concentrated portfolios the expected index becomes very small and the Gini asymptotes to unity.
The Gini is thus an alternative "first moment", similar to the HHI measure we saw above, but in contrast to the HHI, it uses only the *rank order* of the exposures and not the exposure values. Maybe worth mentioning that in analogy with the Hannah-Kay family we can define various higher order moments of the index distribution of I:

$$\mathbb{E}(I^a) = \sum_{i=1}^{n} w_i i^a \tag{20}$$

but their advantages are not immediately obvious.

Shannon (Entropy) Index

In our notation the Shannon Index of a portfolio configuration is defined as

$$S = -\sum_{i=1}^{n} w_i \log w_i \tag{21}$$

The interpretation of the Shannon index is typically in terms of the *entropy* of the distribution. It may be instructive to re-derive this interpretation as it sheds further light on our probabilistic setup: We divide the total portfolio exposure E_T into ν small exposures. The likelihood of having a configuration where ν_1 small exposures are forming exposure E_1, ν_2 are forming E_2, etc. up to ν_n exposures forming E_n is given by the multinomial symbol Ω_ν, expressing all the possible ways of arranging ν objects into n buckets.

$$\Omega_\nu = \frac{\nu!}{\nu_1!\nu_2!\ldots\nu_n!} \tag{22}$$

The general definition of entropy is

$$\begin{aligned} S_\nu &= -\log \Omega_\nu && (23)\\ &= \log \frac{\nu!}{\nu_1!\nu_2!\ldots\nu_n!} && (24)\\ &= \log(\nu!) - \sum_{i=1}^{n} \log(\nu_i!) && (25)\\ &\approx \nu \log \nu - \sum_{i=1}^{n} \nu_i \log \nu_i && (26)\end{aligned}$$

where the last approximation is valid for large numbers of ν (sufficiently sampling small exposures). Upon substituting the portfolio weights

$$\nu_i = w_i\, \nu \tag{27}$$

we obtain

$$S_\nu = -\nu \sum_{i=1}^{n} w_i \log w_i \tag{28}$$

which means that the entropy per unit exposure is

$$S = \frac{S_\nu}{\nu} = -\sum_{i=1}^{n} w_i \log w_i \tag{29}$$

The interpretation of the Shannon index is that it captures how far from a random (high entropy) distribution is our actual portfolio. In a low entropy portfolio distribution, random small exposures will have an unusually large chance of belonging to a few large exposures.
Rewritting the expression in terms of exposure we see that the Shannon index is linked to the *expected log-exposure*

$$S = \log E_T - \sum_{i=1}^{n} w_i \log E_i = \log E_T - \mathbb{E}(\log E) \tag{30}$$

The lowest entropy (highest information) portfolio is when the expected log exposure is equal to the logarithm of the total portfolio exposure.
For comparability with the other indexes we use a normalized entropy

$$S' = 1 - \frac{S}{\log N} \tag{31}$$

Moment Generating and Characteristic Functions

We see thus that all usual concentration indices can be interpreted as standard probabilistic expectations, with the exception of the CR_k index which captures directly the distribution weights.
This generalization leads us naturally to the moment generating functions and characteristic functions for the exposure E and exposure index I respectively:

$$\mathrm{MGF}_E(t) = \mathbb{E}(e^{tE}) = \sum_{i=1}^{n} w_i e^{tE_i} \tag{32}$$

$$\mathrm{CF}_E(t) = \mathbb{E}(e^{jtE}) = \sum_{i=1}^{n} w_i e^{jtE_i} \tag{33}$$

$$\mathrm{MGF}_I(t) = \mathbb{E}(e^{tI}) = \sum_{i=1}^{n} w_i e^{ti} \tag{34}$$

$$\mathrm{CF}_I(t) = \mathbb{E}(e^{jtI}) = \sum_{i=1}^{n} w_i e^{jti} \tag{35}$$

where j is the imaginary number. Essentially all the ”concentration” content of a portfolio distribution is captured (and can be derived) by suitable manipulations of these functions.

Concentration Index Sensitivity to Concentration

When choosing a concentration index for portfolio management applications a significant practical criterion should be the *sensitivity* of said index to changes in concentration. This follows from the fact that most applications involve ongoing monitoring of portfolios, setting limits and thresholds for mitigating actions etc. Hence the ability of an index to detect and highlight concentration changes is an important feature for risk management purposes.

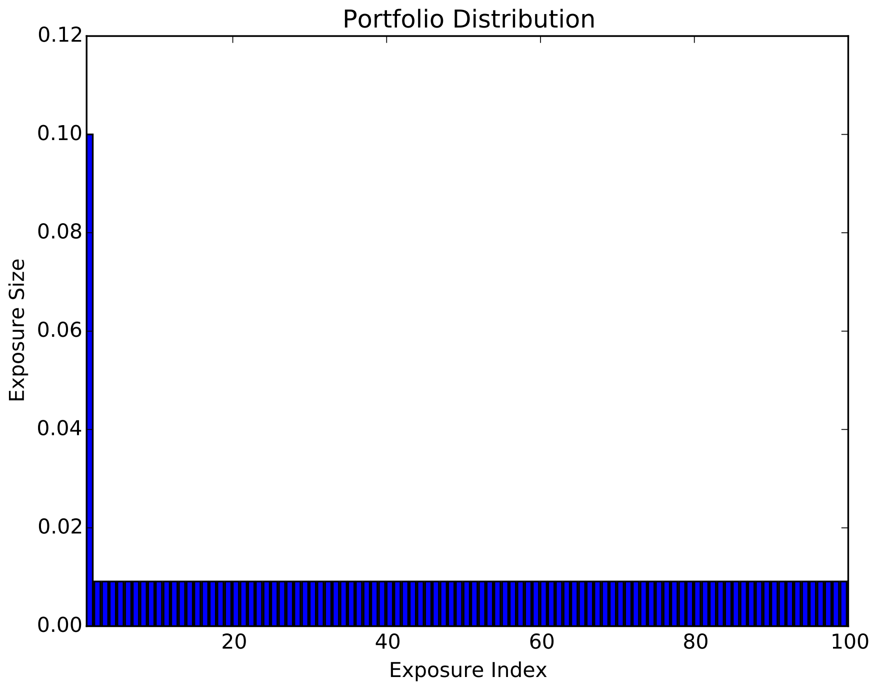

Figure 1: Illustration of a portfolio distribution with a single large exposure where $n = 100$ and $f = 0.1$

It does not seem possible to rank indexes for sensitivity to concentration in completely general terms. Indeed each of the indexes *defines* an aspect of "concentration" hence without some further structure / preferences it would be difficult to derive a general sensitivity ranking.
Therefore we narrow the discussion (thus losing some generality) in order to evaluate the different indexes in more concrete contexts. We study two specific and rather different choices of portfolio distributions. It will transpire that the results are quite robust to the choice [1]

Single Large Exposure

This is an artificial portfolio construct that can capture the most extreme concentration phenomena, namely we set a *single* large exposure (see Fig.1) to a fractional value f,

$$w_1 = f \tag{36}$$

$$w_i = \frac{1-f}{n-1}, \; i \in [2, n] \tag{37}$$

We vary the large exposure and compute all indexes as function of that parameter (see Fig.2) for $n = 100$ and for $n = 1000$ (see Fig.3). We notice the following:

[1] Using an actual (empirical) distribution would not be particularly illuminating for this test, unless we have a sequence of real portfolios that captures concentration increases in a structured manner

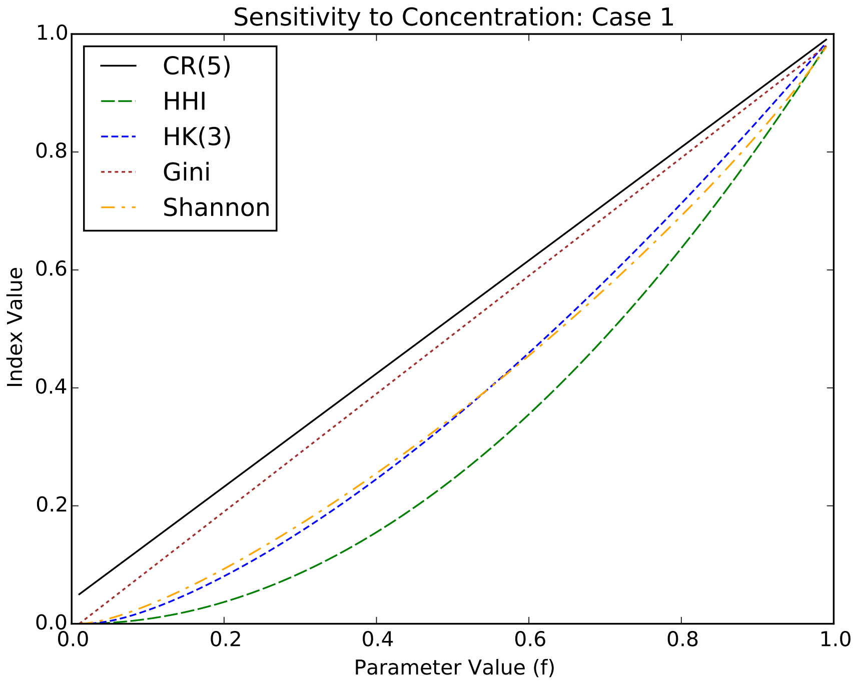

Figure 2: The single large exposure varies from 1% to 100% ($n = 100$)

- The concentration ratio and the Gini are linear in the large exposure. For the concentration ratio this is because it is linear, being simply the sum of large exposures. For the Gini index this is because it only depends on the weight of the largest exposure.
- The other indexes are sub-linear (less sensitive) with the HHI being the least sensitive
- The broad ordering of indexes versus sensitivity does not depend critically on the size of the portfolio, but the details are affected. For example the Shannon index is more sensitive than then third-order Hannah-Kay index for larger portfolios

Power Law Distribution

Turning into a less extreme example of portfolio concentration, we imagine that there is an underlying *portfolio generating process* and hence that the exposures in the portfolio follow a particular law. We consider the case where the exposure distribution follows *Zipf's* law, namely distributed as a power law. This distribution is interesting as it can generate large exposure concentrations,is conveniently parametrized by a single power law index a and the fact that power laws are typical in many economic phenomena. Specifically we assume that

$$w_i = \frac{1}{H_{n,a}} \frac{1}{i^a} \tag{38}$$

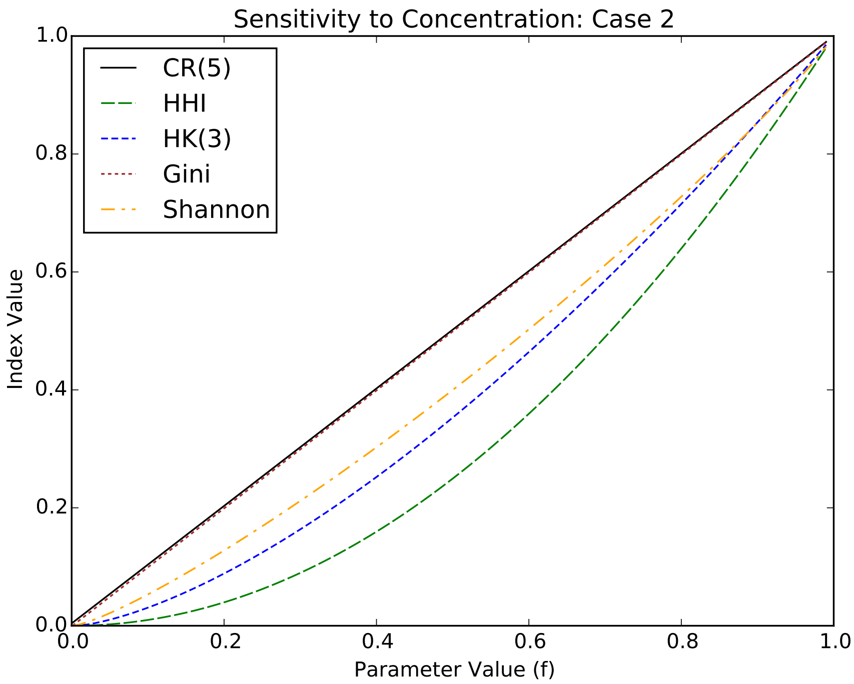

Figure 3: The single large exposure varies from 1% to 100% ($n = 1000$)

where $H_{n,a}$ is the generalized harmonic number

$$H_{n,a} = \sum_{i=1}^{n} \frac{1}{i^a} \tag{39}$$

With this simplification, the propensity of a portfolio to have proportionally more large exposures is captured by the single power law index a. Examples of the distribution for two indicative values of a are given in Fig.4. Note that the plot is in log-log scale to bring out the power law behavior.

In the plot (see Fig.5) we illustrate the variation of the various indexes as the parameter a of the distribution changes. When focusing on the slope of the curves we note the following:

- The most sensitive index (highest slope) is the CR, followed by the Gini and Shannon indexes
- The HHI is again the least sensitive
- The HK/HHI family has a brief region around low concentration where sensitivity increases (but starting from a lower baseline value)

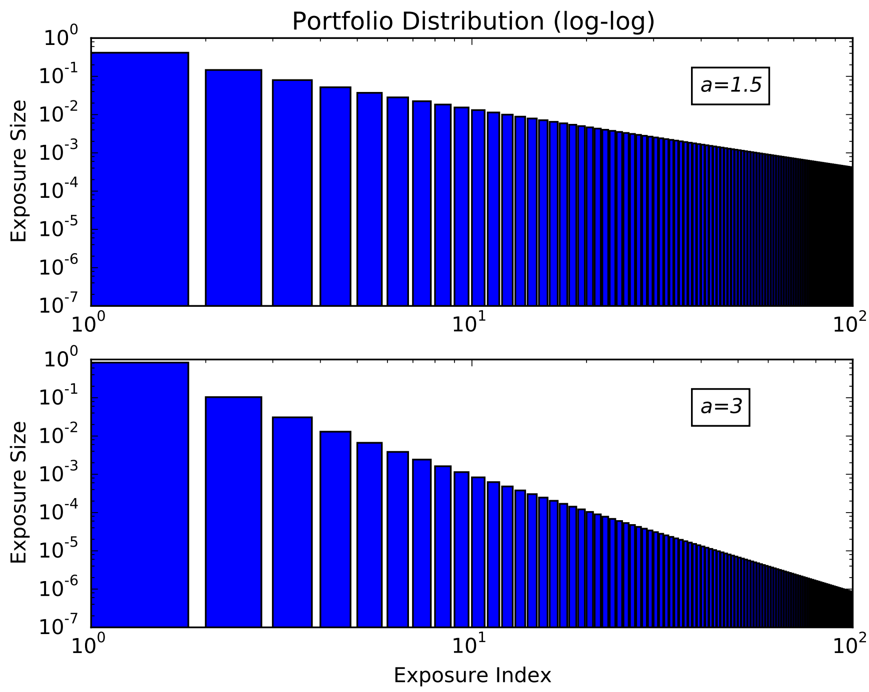

Figure 4: Zipf (power law) portfolio distribution for two values of a ($n = 100$).

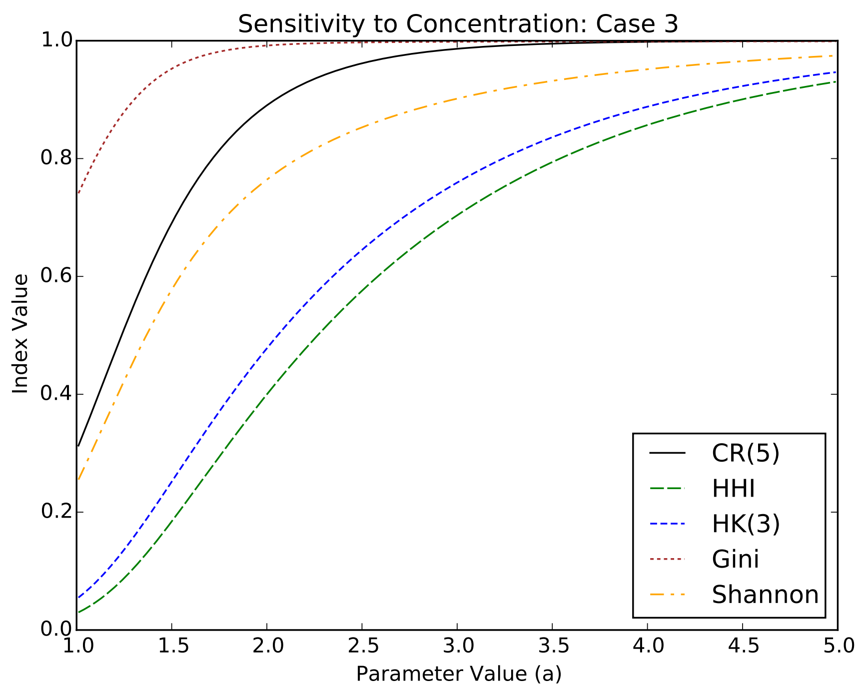

Figure 5: The power law parameter a varies between 1 and 5 ($n = 100$)

Open Source Implementation

A software implementation and related documentation of the metrics developed in this paper are available and released by OpenRisk under Open Source and Creative Commons licenses respectively.

Requirements

To use the implementation you will need

- A functioning python installation on any platform (available for all major platforms, see `www.python.org`)
- The following additional python libraries: {numpy, scipy}. Check the respective websites for installation instructions (`www.numpy.org`, `www.scipy.org`)
- The OpenRisk library from our Github repository (`www.github.com/open-risk`)
- Portfolio data in a simple ascii file

Documentation of available functions

Documentation for the available functions is provided in the OpenRisk Manual:

- The Concentration Ratio
- The HHI index
- The Gini index
- The Shannon index
- The Hannah-Kay index

Code Structure and Usage

The library consists of a single python module that can be imported and used in other projects. Consult the Readme for usage and testing instructions

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